Appendix A

Sundry mathematical ideas

A.1 Plane and solid angles (Chapters 2, 4, 7 and 8)

A **plane** angle θ in radians (Fig. A.1) is given by the ratio s/r, where s is the length of the arc of a circle centred at O cut off by the arms of the angle and r is the radius of this circle. For a small angle $d\theta$, the chord and arc differ in length by a second order of smallness and $d\theta$ is given equally well by ds/r or dl/r.

A **solid** angle Ω is an extension to three dimensions of the above idea. To define a solid angle (Fig. A.2), a sphere of radius r is constructed with centre at the apex of the angle O. Suppose the area cut off on the surface of this sphere by the generators of the angle is S. The solid angle is given in *steradians* by

$$\Omega = S/r^2 \qquad \text{(definition of } \Omega\text{)} \tag{A.1}$$

It follows that the complete solid angle about a point is 4π because the surface area of a sphere is $4\pi r^2$. Another useful result which also follows from the definition is that the solid angle at the apex of a cone of semi-vertical angle θ is

$$\Omega_{\text{cone}} = 2\pi(1-\cos\theta) \tag{A.2}$$

For a small solid angle $d\Omega$, the plane area dA differs from dS by a second order of smallness so that $d\Omega = dS/r^2$ of dA/r^2. If an area dS is not normal to the radius r, then it must be projected on to a plane which *is* normal to r as an area $dS\cos\theta$. The solid angle subtended by it at O is then given by

$$d\Omega = \frac{dS\cos\theta}{r^2} \tag{A.3}$$

A.2 Coordinate systems, particularly polar coordinates (Chapter 3 onwards)

Two dimensions Instead of specifying the position of a point in a plane by its Cartesian coordinates x and y, it is sometimes convenient to use plane polar coordinates (r, θ). Figure

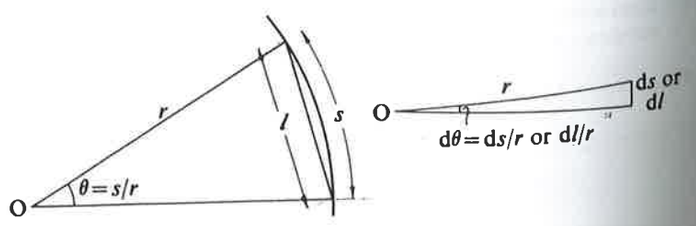


Figure A.1 Plane angles: θ and $d\theta$ are in radians.

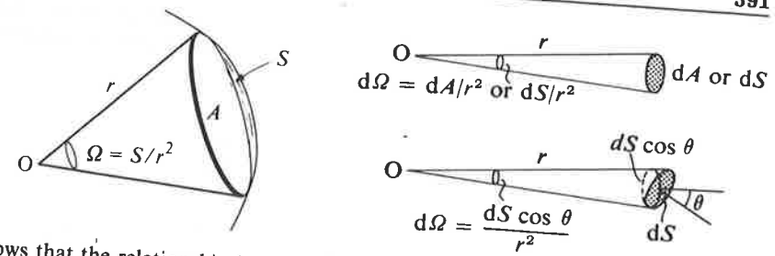


Figure A.2 Solid angles: Ω and $d\Omega$ are in steradians.

A.3 shows that the relationship between the two sets will be

$$x = r\cos\theta; \qquad y = r\sin\theta \tag{A.4}$$

or

$$r = (x^2+y^2)^{1/2}; \qquad \theta = \tan^{-1}(y/x) \tag{A.5}$$

Three dimensions To obtain the **Cartesian coordinates** (x, y, z) of a point in three-dimensional space, a z axis is added at right angles to the x and y axes. The positive direction of the z axis can be chosen in one of two senses (up or down in Fig. A.4, in which the x axis points out of the page). It is conventional to choose the *right-handed* system illustrated in Fig. A.4, so called because the rotation of a right-handed screw from the x direction to the y direction about z as axis would cause the screw to move along z. A left-handed system would have the z axis pointing downwards.

Polar coordinates are sometimes simpler to use than Cartesians. The **cylindrical polar coordinates** (r, θ, z) of a point are obtained by using a z axis as in Cartesians but adding it to plane polar (r, θ) coordinates in the xy plane as shown in Fig. A.4a. The relationships between (r, θ, z) and (x, y, z) are as in Eqs (A.4) and (A.5) above, the z's being the same.

Alternatively, the position of a point can be expressed in **spherical polar coordinates** (r, θ, ϕ) defined in relation to Cartesians as shown in Fig. A.4b. The relationships between (r, θ, ϕ) and (x, y, z) are

$$x = r\sin\theta\cos\phi; \qquad y = r\sin\theta\sin\phi; \qquad z = r\cos\theta \tag{A.6}$$

or

$$r = (x^2+y^2+z^2)^{1/2}; \qquad \theta = \cos^{-1}(z/r); \qquad \phi = \tan^{-1}(y/x) \tag{A.7}$$

Note that the r in spherical polars is the distance from the *origin*, while in cylindrical polars it is the *perpendicular distance from the z axis*.

Which of the coordinate systems is best often depends on the symmetry of the situation. For instance, the potential due to a point charge at the origin depends only on the distance from the charge so that it can be written most simply in spherical polars as $V = Q/(4\pi\varepsilon_0 r)$, there being no dependence on θ or ϕ. In Cartesians, this would become $V = Q/[4\pi\varepsilon_0(x^2+y^2+z^2)^{1/2}]$, a more cumbersome form that would only be used if a subsequent problem demanded the use of x, y, z.

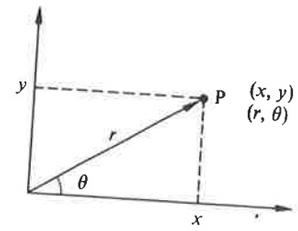


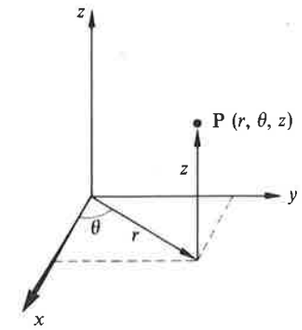
Figure A.3 Plane Cartesian and plane polar coordinates.

Small displacements, areas and volumes In many situations encountered in this book, small increments dL in displacement are used to isolate a typical element of a path in space or of a length of an object. In two dimensions, such an element is in general a combination of two orthogonal increments along the principal axes—in Cartesians these increments would be dx and dy. In plane polars, the orthogonal increments are dr (radially) and $r\,d\theta$ (tangentially, to the radial coordinate), as shown in Fig. A.5.

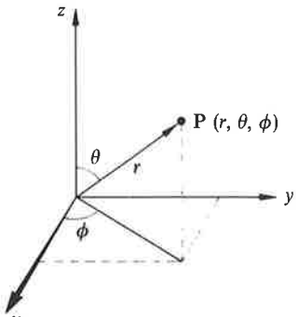
In three dimensions, the orthogonal increments are

$$\begin{aligned} &dx,\ dy,\ dz\text{—Cartesians} \\ &dr,\ r\,d\theta,\ dz\text{—cylindrical polars} \\ &dr,\ r\,d\theta,\ r\sin\theta\,d\phi\text{—spherical polars} \end{aligned} \tag{A.8}$$

[These are in fact the components of an elementary vector displacement—see Sec. B.8 of Appendix B and the gradient in polars given in Eqs (B.30) and (B.34).]



(a)



(b)

Figure A.4 (a) Cylindrical polar coordinates and their relationship to Cartesians; (b) spherical polar coordinates and their relationship to Cartesians.

Elementary areas and volumes are also used and are constructed from the increments in (A.8). Thus a typical elementary area in the Cartesian xy plane is $dx\,dy$, while typical volume elements $d\tau$ in the three systems are

$$\begin{aligned} d\tau &= dx\,dy\,dz && \text{—Cartesians} \\ d\tau &= r\,dr\,d\theta\,dz && \text{—cylindrical polars} \\ d\tau &= r^2 \sin\theta\,dr\,d\theta\,d\phi && \text{—spherical polars} \end{aligned} \qquad \text{(A.9)}$$

A.3 Partial differentiation, partial derivatives (Chapter 3 onwards)

When a quantity Z is a well-behaved† function of one other quantity x, it can be represented by a curve $Z=f(x)$ as in Fig. A.6a: dZ/dx can be evaluated at any point such as P, and is equal to the slope of the tangent to the curve at that point.

If, however, Z is a function of two quantities, x and y, then $Z=f(x,y)$ is a *surface* in x, y, Z coordinates, part of which is drawn in Fig. A.6b. In this case, dZ/dL can have an infinite number of values at a point P depending on the direction of L in the xy plane. An example is the rate of change of height of a hill with distance: the result depends entirely on the direction of travel.

When we have $Z=f(x,y)$ we commonly use two rates of change—that of Z with x when y is held constant, denoted by $(\partial Z/\partial x)_y$, and of Z with y when x is held constant, $(\partial Z/\partial y)_x$. In Fig. A.6b these correspond to the slopes at P and Q of sections of the surface cut by xZ and yZ planes respectively. They are known as *partial differential coefficients* or *partial derivatives*.

We often wish to know the increment in Z when both x and y vary by dx and dy, respectively. Since the increment due to dx alone is $(\partial Z/\partial x)_y\,dx$ and that due to dy alone is $(\partial Z/\partial y)_x\,dy$, the total differential dZ is

$$dZ = \left(\frac{\partial Z}{\partial x}\right)_y dx + \left(\frac{\partial Z}{\partial y}\right)_x dy \qquad \text{(A.10)}$$

For functions of more than two variables, say $Z=f(x,y,z)$, the use of $(\partial Z/\partial x)$ implies that all the other variables are to be held constant. For instance, if $Z=x^2y+2y^2+3z^2$, then $\partial Z/\partial x=2xy$; similarly, if $Z=xy+yz$ then $\partial Z/\partial x=y$. The function Z cannot now be drawn but the total increment in Z due to increments in x, y and z is, by an extension of (A.10),

$$dZ = \left(\frac{\partial Z}{\partial x}\right)dx + \left(\frac{\partial Z}{\partial y}\right)dy + \left(\frac{\partial Z}{\partial z}\right)dz \qquad \text{(A.11)}$$

As an example, the potential V of Chapter 3 is *in general* a function of three variables because its value depends on position in space. In Cartesians it would be a function of x, y and z, and its three principal gradients would be denoted by $\partial V/\partial x$, $\partial V/\partial y$, $\partial V/\partial z$.

Occasionally a quantity may be expressed in terms of one set of coordinates and the variation with respect to another is wanted. For instance, the potential V due to a dipole is usually in plane polar coordinates and $\partial V/\partial x$ may be required. In carrying out such a calculation, expressions like $\partial r/\partial x$, $\partial\theta/\partial x$, etc., arise. They are easily evaluated when needed from Eqs (A.4) and (A.5). For example,

$$\begin{aligned} \partial x/\partial r &= \cos\theta = x/r; & \partial x/\partial\theta &= -r\sin\theta = -y \\ \partial r/\partial x &= x/r = \cos\theta; & \partial\theta/\partial x &= -y/r^2 = -(\sin\theta)/r \end{aligned} \qquad \text{(A.12)}$$

with corresponding expressions for y. Note particularly that $\partial x/\partial r \neq 1/(\partial r/\partial x)$.

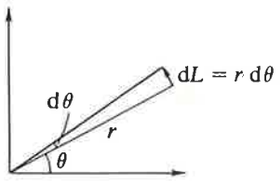


Figure A.5 An element of displacement perpendicular to the radial coordinate.

† We usually assume *at least* that the function and its first derivative are single-valued continuous functions of x.

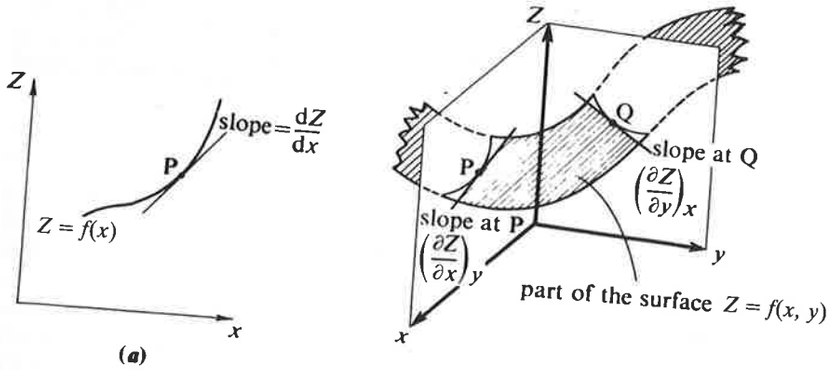


Figure A.6 (a) The line $Z=f(x)$; (b) the surface $Z=f(x,y)$.

A.4 Line integrals (Chapter 2 onwards)

I find the best way to understand line integrals is to see them as generalizations of ordinary definite integrals like $\int_1^2 x^2\,dx$. Instead of interpreting this as an area under a $y=x^2$ curve, think of it as a summation along the x axis: each element of the x axis, dx, has a value x^2 associated with it. Form the product $x^2\,dx$ for that element and sum all such products from $x=1$ to $x=2$. This is in fact a line integral taken along the x axis.

To generalize the idea to any path, suppose that a scalar quantity F has a value at every point in a region of space. It will therefore have a value at every point along a line L in that region between the points A and B (Fig. A.7a). For a small element of the path dL form the product of dL with the value of F at the element: $F\,dL$. The *line integral* of F between A and B along L is the sum of all the $F\,dL$'s from A to B and is written $\int_A^B F\,dL$.

Line integrals occur most frequently when the quantity varying along the line is a vector $\mathbf{F}$, for instance a force or electric field. In that case, the line integral is formed by taking the product of $d\mathbf{L}$ with the resolved part of $\mathbf{F}$ along $d\mathbf{L}$, $F\cos\theta$, and summing as before. Thus the work done by a force F along the path L from A to B is $\int_A^B F\cos\theta\,dL$. (This could be written as $\int \mathbf{F}\cdot d\mathbf{L}$ using the notation for a scalar product as in Appendix B.2.)

The evaluation of line integrals is a matter best left to mathematical texts, but it is evident that the result will in general depend both on the positions of A and B and on the path L. In certain cases involving vector quantities, however, the line integral is independent of the path. For instance, we saw in Sec. 2.7 that, if $\mathbf{F}$ is a central force, then

$$\int_A^B F\cos\theta\,dL = \int_{r_A}^{r_B} F(r)\,dr = f(r_B) - f(r_A) \qquad \text{(A.13)}$$

where r is the radial distance from the centre of force and $f(r)$ is the indefinite integral of $F(r)$, and this depends only on A and B. The physical reason for the path-independence has been discussed in Sec. 2.7. *Where path-independence occurs, we choose a path for integration which makes the integration as simple as possible.*

Sometimes a line integral round a closed path (as in Fig. A.7b) is required and it is conventional to indicate this by the symbol $\oint$. For a vector quantity $\mathbf{F}$, $\oint \mathbf{F}\cdot d\mathbf{L}$ is called the *circulation* of $\mathbf{F}$ round the path. When a function of position F has a *path-independent* line integral, $\oint F\,dL$ is zero since A and B in Eq. (A.13) coincide.

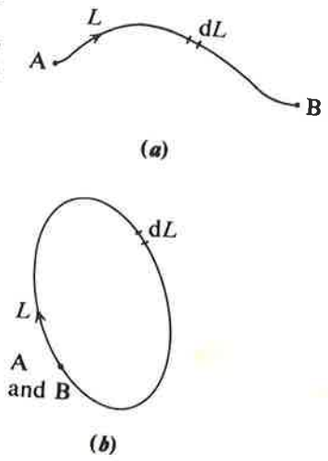


Figure A.7 (a) An open path along which the line integral of any function F of position can be evaluated; (b) a closed path: A and B now coincide.